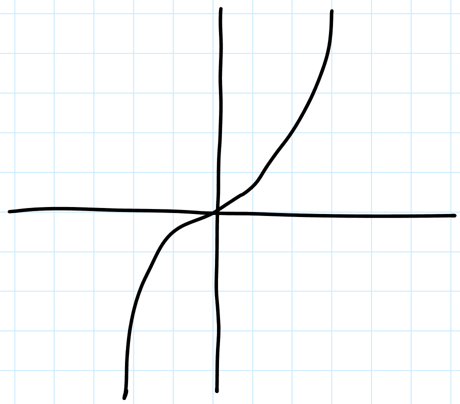
Curvature how much the path of a vector-valued function "curves"
- zero iff the path is a line
- why not 2^{nd} derivative?

Consider $y = x^3$ (in param: $(t, t^3) = (x, y)$)



2^{nd} deriv $6x$

So, as x gets large, so does the 2^{nd} deriv

Curvature?
Gets small as x gets really large bc path becomes really close to being a vertical line

Idea 2^{nd} derivative (acceleration)
- measures change in velocity
Two ways velocity can change:
1. change direction } → curvature
2. change speed

Given $\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$\vec{r}\,' = \frac{d\vec{r}}{dt} = \vec{v}$ is velocity

speed: $\|\vec{v}\|$
direction: $\frac{\vec{v}}{\|\vec{v}\|} = \vec{T}$ ← unit tangent vector

try $\frac{d\vec{T}}{dt}$

Problem this depends on parameterization
try $\frac{d\vec{T}}{ds}$

Problem this is a vector

Try
$\underbrace{\left\|\frac{d\vec{T}}{ds}\right\|}_{\kappa}$ ← this is curvature!

κ = kappa

What about direction of $\frac{d\vec{T}}{ds}$?

Define $\vec{N} = \frac{d\vec{T}/ds}{\|d\vec{T}/ds\|}$

so $\frac{d\vec{T}}{ds} = \kappa \vec{N}$

See [Co] Sec. 1.9, prob 10 for κ in terms of $\vec{r}(t)$ (aka $\vec{r}(t)$)

Why "$\vec{N}$"?

Because: $\vec{T}\cdot\vec{T} = 1$ is const

if we d/ds both sides

$$2\vec{T}\cdot\frac{d\vec{T}}{ds} = \frac{d}{ds}\vec{T}\cdot\vec{T} = \frac{d}{ds}1 = 0$$

So $\vec{T}\cdot\frac{d\vec{T}}{ds} = 0$ so $\frac{d\vec{T}}{ds}$ is perpendicular to $\vec{T}$

Similarly, since $\vec{N} = \frac{d\vec{T}/ds}{\text{scalar}}$ is in the same direction as $d\vec{T}/ds$, $\vec{N}$ is also perpendicular to $\vec{T}$.

→ N stands for Normal

"unit normal vector"

So $\vec{N}$ is the direction in $\vec{T}$ is changing

Now, in 3-space ($\mathbb{R}^3$), we can consider the plane spanned by $\vec{T}$ and $\vec{N}$

"the plane in which the object is infinitesimally moving in"

- so if $\vec{r}(t)$ stays in that plane, T & N are in that plane
- if $\vec{r}(t)$ doesn't stay in a plane, then the plane spanned by $\vec{T}$ and $\vec{N}$ changes over time

Torsion (τ) = measure of how much the plane is changing

How to measure?

Define $\vec{B} = \vec{T}\times\vec{N}$

Notice since $\|\vec{T}\| = \|\vec{N}\| = 1$ and $\vec{T}\cdot\vec{N} = 0$, also $\|\vec{B}\| = 1$

By considering $\frac{d}{ds}(\vec{B}\cdot\vec{B})$, we find that $\frac{d\vec{B}}{ds}$ is $\perp$ to $\vec{B}$

rough idea τ = torsion

$$= \|d\vec{B}/ds\|$$

Problem want to allow τ to be negative

Better idea

Notice $\frac{d\vec{B}}{ds}$ is $\perp$ to $\vec{B}$ and to $\vec{T}$ => parallel to $\vec{N}$.

$$\therefore \frac{d\vec{B}}{ds} = (\text{scalar})\cdot\vec{N}$$

Define τ by

$$\frac{d\vec{B}}{ds} = -\tau \vec{N}$$

See 4.3 [CH]

eg $\tau \neq 0$ for a helix

Why is $\frac{d\vec{B}}{ds} \perp \vec{T}$?

$\vec{B} \cdot \vec{T} = 0$ d/ds both sides

$$\frac{d\vec{B}}{ds} \cdot T + B \cdot \frac{d\vec{T}}{ds} = 0$$

and $\frac{d\vec{T}}{ds} \parallel \vec{N}$ so $B \cdot \frac{d\vec{T}}{ds} = 0$

* Ignore Maple calculations in [CH] *

Functions of Multiple Variables [Co] 2.1

e.g.

$f(x,y) = xy$ defined $\forall (x,y) \in \mathbb{R}^2$

$f(x,y) = \frac{1}{x-y}$ defined for only some $(x,y) \in \mathbb{R}^2$

Defined on some subset $D \subseteq \mathbb{R}^2$

→ in this case when $x \neq y$

So $D = \{(x,y) \in \mathbb{R}^2 \mid x \neq y\}$

set of (x,y) in $\mathbb{R}^2$ — such that

$= \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 \mid x = y\}$

"points in $\mathbb{R}^2$ not on the line $x = y$"

Defn A real-valued fcn $f(x,y)$ assigns a real # to every $(x,y) \in D \subseteq \mathbb{R}^2$

↳ D is domain of f

(If n-vars, $f(x,y\ldots,n)$ and $D \subseteq \mathbb{R}^n$)

Back to 2 vars:

Geometrically its graph is a surface in $\mathbb{R}^3$ (Just like graph of $y = f(x)$ is a curve in $\mathbb{R}^2$)

The points of the graph are $(x,y,f(x,y))$ for $(x,y) \in D$.

Level curves Given $f(x,y)$ and $c \in \mathbb{R}$, the level curve is the set of $(x,y) \in D$ such that $f(x,y)=c$.

In set notation: $\{(x,y) \in D \mid f(x,y)=c\}$

Notice: if f is a const fcn

eg $f(x,y)=4$

The level curve is:

$\emptyset$ (empty set) if $c \neq 4$

$\mathbb{R}^2$ (whole plane) if $c=4$

egs Where it is a curve

- $f(x,y)=3x-2y$

  Then all the level curves are lines perpendicular $(3,-2)$

  [As you change c, you get diff lines, but all are $\parallel$ to each other]

- $f(x,y)=x^2+y^2$

  the level curve is a circle if $c>0$

  point if $c=0$

  $\emptyset$ if $c<0$

For any single variable func g, set $f(x,y)=y-g(x)$

Then level curve w/ $c=0$ is graph of g

Note: Level curves are traces of the graph of $z=f(x,y)$ on horizontal planes

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Limits & Continuity

Limits say $f(x,y)$ defined "near (a,b)" but not necessarily at (a,b)

Formally suppose $f(x,y)$ is def'd in a "punctured neighborhood" of (a,b)

i.e. a set of the form

$$\{(x,y) \in \mathbb{R}^2 \mid \underset{\uparrow}{0 <} \|(a,b)-(x,y)\| < \varepsilon\}$$

"punctured"

for some $\varepsilon > 0$

We say:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if $\forall \varepsilon > 0 \quad \exists \delta > 0$

such that if $\|(x,y)-(a,b)\| < \delta$ but $(x,y) \neq (a,b)$

then

$|f(x,y) - L| < \varepsilon$

Intuitively as (x,y) gets closer to (a,b), $f(x,y)$ gets closer to L.

Caveat must be true no matter which direction (x,y) approaches (a,b)

e.g.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \quad \text{DNE}$$

Why?

If you approach along x or y axis, then it seems the limit is 0 bc $f(x,y)$ gets closer & closer to 0

BUT if $(x,y) \to (0,0)$ along $y = x$, then $f(x,y)$ approaches $1/2$

$f(x,y) = \sin\theta\cos\theta$ for (θ, r) polar coords

Basic properties of limits are the same (addition, sub, mult, div)

↳ as long as denom does not approach 0.

Continuity

Suppose $f(x,y)$ is defined for (x,y) near (a,b) incl. at (a,b) itself if $f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$

As in single-var calc, sums, products, quotients (if denom $\neq 0$) of continous fcns are continuous

BUT, if denom $= 0$, you may/may not be able to make it cont at (a,b)

- $f(x,y) = \dfrac{xy}{x^2+y^2}$ can't make it cont at $(0,0)$
- $f(x,y) = \begin{cases} \dfrac{y^4}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \end{cases}$

$$\bullet \; f(x,y) = \begin{cases} \dfrac{y^4}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

is continuous.